Geometric Series

The geometric is a very important series in mathematics. The series is written as:

s = $\overset{\infty}{\underset{n=0}{}}\sum a*r^n$ or s=$\overset{\infty}{\underset{n=1}{}}\sum a*r^{(n-1)}$

Here 's' is equal to the sum of the series and can be found by:

s = $a+a*r+a*r^2+a*r^3+....+a*r^{n-1}$

$r*s=a*r+a*r^2+a*r^3+....+a*r^{n-1}+a*r^n$

$s-r*s=a\ -a*r^n$

$s(1-r)=a(1-r^n)$ and $\lim_{n\to\infty} r^n=0 \text{ if } r<1$

s = $\frac{a}{1-r}$ when r<1

Now we can write $\frac{a}{1-r}=\overset{\infty}{\underset{n=0}{}}\sum a*r^n$ and this series is convergent for all r<1 where 'a' is arbitrary constant.

Power Series

One of the applications of a geometric series is that we can use it to represent functions as a power series. So if we can manipulate a function to get it in the form of $\frac{1}{1-x}$ it can be written as a power series. In general, $\frac{1}{1-x}=\overset{\infty}{\underset{n=0}{}}\sum x^n$.

Example:

Seeing how things work always help me to visualize things more so here is an example of how a function can be written as a power series.

$f(x)=\int \frac{dx}{1+x^3}$

The first step is to write $\frac{1}{1+x^3}$ as a power series:

$\frac{1}{1+x^3}=\frac{1}{1-(-x^3)}=\overset{\infty}{\underset{n=0}{}}\sum (-x^3)^n=\overset{\infty}{\underset{n=0}{}}\sum (-1)^n x^{3n}$

The last thing we need to do to get f(x) is to integrate the series:

f(x) = $\overset{\infty}{\underset{n=0}{}}\sum (-1)^n \int x^{3n}\,\mathrm{dx}=\overset{\infty}{\underset{n=0}{}}\sum (-1)^n \frac{x^{3n+1}}{3n+1}$

Taylor and Maclaurin Series

A taylor series is similar to a power series, except that the coefficients are variable.

f(x) = $c_0 + c_1*x + c_2*x^2 + c_3*x^3 + ...$

$f'(x) = c_1 + 2*c_2*x + 3*c_3*x^2 + ...$

$f''(x) = 2*c_2 + 6*c_3*x + ...$

$f'''(x)=6*c_3 + ...$

We can see a pattern developing so now we can find c_n, the nth coefficient:

$f^{(n)}(x)=n!*c_n$ This can be rearranged to find c_n.

$c_n = \frac{f^{(n)}(x)}{n!}$

In general, any function can be written as a taylor series centered at 'a':

f(x) $= {}_{n=0}^{\infty}\sum \frac{f^{(n)}(x)}{n!}$(x-a)^n

You may be wondering how we got (x-a) in there and its actually very simple. Remember the rules for translation of functions. A function shifted to the right is f(x-a). If a = 0, then the function is centered about zero, and this type of series is a Maclaurin series.

An Example:

Just so you can visualize what all this means I'm going to show how you can find the series for e^x. After you find a series you need to find the radius of convergence, which I am not going to discuss, but this can be done with the ratio test.

f(x) = e^x The function for which we want to find the series is the exponential function.

Let's center the series at 0 so we are finding the Maclauring series.

f(0) =1

f$^{(n)}(x) = e^x$ so the nth derivative of e^x is e^x. This leads us to the conclusion that f$^{(n)}(0) = 1$.

f(x) $= {}_{n=0}^{\infty}\sum \frac{f^{(n)}(x)}{n!}$(x-a)^n $= 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + ... = {}_{n=0}^{\infty}\sum \frac{x^n}{n!}$

Common Maclaurin Series:

exp(x) $= {}_{n=0}^{\infty}\sum \frac{x^n}{n!}$ Converges for all x.

$\sin(x) = {}_{n=0}^{\infty}\sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ Converges for all x.

$\cos(x) = {}_{n=0}^{\infty}\sum (-1)^n \frac{x^{2n}}{(2n)!}$ Converges for all x.